Journal of Approximation Theory **102**, 72–95 (2000)
Article ID jath.1999.3378, available online at http://www.idealibrary.com on IDEAL®

Markov–Bernstein Type Inequalities for Multivariate Polynomials on Sets with Cusps

A. Kroó and J. Szabados[1]

Alfréd Rényi Mathematical Institute, Hungarian Academy of Sciences, 1364 Budapest, P.O.B. 127, Hungary

Communicated by Rong Qing Jai

Received June 24, 1998; accepted in revised form March 25, 1999

In this paper we give sharp Markov–Bernstein type inequalities for derivatives of multivariate polynomials for a wide family of domains with cusps.

Key Words: multivariate polynomials; derivatives; domains with cusps; polynomial curves.

1. INTRODUCTION

Markov–Bernstein type inequalities relating the magnitude of norms of polynomials to their derivatives play an important role in many areas of mathematical research. The classical Markov inequality for univariate polynomials states that for any polynomial $p_n(x)=\sum_{j=0}^{n} a_j x^j$ $(a_j \in \mathbf{R})$ of degree at most n

$$\|p_n'\|_{C[a,b]} \leqslant \frac{2n^2}{b-a}\|p_n\|_{C[a,b]}. \tag{1}$$

The magnitude of the derivative may be substantially smaller inside the interval (a, b) which is reflected in the Bernstein inequality

$$|p_n'(x)| \leqslant \frac{n}{\sqrt{(x-a)(b-x)}}\|p_n\|_{C[a,b]}, \qquad x \in (a, b). \tag{2}$$

In the last twenty years possible extensions of the above estimates for multivariate polynomials have been widely investigated. In this context a compact set $\mathbf{K} \subset \mathbf{R}^m$ with nonempty interior is given and the space

$$\mathscr{P}_n^m := \left\{ \sum_{|\mathbf{k}| \leqslant n} a_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} : a_{\mathbf{k}} \in \mathbf{R} \right\}, \qquad \mathbf{x} \in \mathbf{R}^m,$$

[1] Research supported by OTKA Grants No. T023441 and T017425.

0021-9045/00 $35.00

of real polynomials of total degree $\leqslant n$ and m variables is considered on $\mathbf{K}$, endowed with the usual supremum norm

$$\|p_n\|_{C(\mathbf{K})} := \max_{\mathbf{x} \in \mathbf{K}} |p_n(\mathbf{x})|.$$

Furthermore, setting

$$Dp_n(\mathbf{x}) := \left(\sum_{j=1}^{m} \left(\frac{\partial p_n(\mathbf{x})}{\partial x_j} \right)^2 \right)^{1/2}$$

for $\mathbf{x} := (x_1, ..., x_m) \in \mathbf{R}^m$, we can consider the nth order *local Markov–Bernstein factor* (shortly MB factor) of $\mathbf{K}$ at $\mathbf{x}$ given by

$$M_n(\mathbf{K}, \mathbf{x}) := \sup\{ Dp_n(\mathbf{x}) \colon p_n \in \mathscr{P}_n^m, \|p_n\|_{C(\mathbf{K})} \leqslant 1\}.$$

As we have seen above, even in the univariate case the size of the derivatives is closely connected to the location of the point considered: it is of magnitude n inside the interval, and n^2 at the endpoints. In the multivariate case the local geometry of the domain becomes even more crucial; we shall see below its dramatic effect on the size of the MB factor. This justifies the above definition of the local MB factor.

Remark. Throughout this paper we shall give estimates for $M_n(\mathbf{K}, \mathbf{x})$ which can be easily extended to uniform MB-factors defined as

$$M_n(\mathbf{K}) := \sup_{\mathbf{x} \in \mathbf{K}} M_n(\mathbf{K}, \mathbf{x}).$$

This extension follows automatically by imposing the needed geometric conditions not only at a fixed $\mathbf{x} \in \mathbf{K}$ but uniformly for every $\mathbf{x} \in \mathbf{K}$ (see Corollary 1 below). We chose to deal with the local MB-factors only in order to keep the notations simpler, but this does not restrict the generality of our considerations.

In what follows we shall assume that $\mathbf{K}$ is a compact set with nonempty interior such that $\mathbf{K} = \overline{\operatorname{int} \mathbf{K}}$, i.e., every point of $\mathbf{K}$ is a cluster point of its interior. Clearly, by (2) we have $M_n(\mathbf{K}, \mathbf{x}) \leqslant c_1 n$ whenever $\mathbf{x} \in \operatorname{int} \mathbf{K}$. Moreover, when $\mathbf{x} \in \operatorname{bd} \mathbf{K}$ (the boundary of $\mathbf{K}$) and $\operatorname{bd} \mathbf{K}$ is Lip 1 at $\mathbf{x}$ we obtain by (1) that $M_n(\mathbf{K}, \mathbf{x}) \leqslant c_2 n^2$. (Here and in what follows, $c_1, c_2, ...$ stand for constants independent of n.) In particular, if $\mathbf{K}$ is locally convex at $\mathbf{x} \in \operatorname{bd} \mathbf{K}$ (i.e., there exists a ball $\mathbf{B}$ centered at $\mathbf{x}$ such that $\mathbf{B} \cap \mathbf{K}$ is convex) then $M_n(\mathbf{K}, \mathbf{x}) \sim n^2$. The upper bound in the latter statement follows from (1) while the lower bound is a special case of Theorem 4 below.

The situation becomes more complex for those boundary points $\mathbf{x}$ of $\mathbf{K}$ where $\operatorname{bd} \mathbf{K}$ is not Lip 1, i.e., when $\mathbf{x}$ is a *cusp*. The first example illustrating the behavior of the MB factor at a cuspidal point was given by Goethgeluck [5] for the domain

$$\mathbf{K}_p := \{(x, y) \in \mathbf{R}^2 : 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant x^p\}, \qquad p > 1$$

(see Fig. 4). This domain has a cusp at the origin, and it was shown in [5] that $M_n(\mathbf{K}_p, \mathbf{0}) \sim n^{2p}$. Thus the MB factor can increase substantially at a cuspidal point. In [6] and [7] the authors considered MB factors for general cuspidal regions. The basic assumption was that the cusp $\mathbf{x} \in \operatorname{bd} \mathbf{K}$ is connected to $\operatorname{int} \mathbf{K}$ by a polynomial curve. That is, there exists a curve $\mathbf{q}(t) = \{l_j(t)\}_{j=1}^m : [0, 1] \to \mathbf{R}^m$ such that $l_j \in \mathscr{P}_r^1$ $(1 \leqslant j \leqslant m)$ for some $r \in \mathbf{N}$, $\mathbf{q}(0) = \mathbf{x}$ and $\mathbf{q}(t) \subset \operatorname{int} \mathbf{K}$ for $0 < t \leqslant 1$. In this case there exists a positive function $w(t)$, $0 < t \leqslant 1$, such that the m-dimensional ball with center at $\mathbf{q}(t)$ and radius $w(t)$ is contained in $\mathbf{K}$. This function w measuring the width of $\mathbf{K}$ around the cusp $\mathbf{x}$ was used for estimating $M_n(\mathbf{K}, \mathbf{x})$. However, the estimates given in [6] and [7] for MB factors are not optimal, even Goetgheluck's example of [5] cannot be recovered by general methods used in [6] and [7]. Recently, Baran [1] gave a sharp estimate $M_n(\mathbf{K}) = O(n^{2p})$ for polynomial cusps with width function $w(t) = t^p$ $(p > 1)$. His approach relied heavily on the theory of plurisubharmonic functions.

The main objective of this paper is to give a further systematic study of the rate of MB factors for cuspidal domains. We provide asymptotically sharp bounds for polynomial cusps with *arbitrary* width functions $w(t)$. The rate of MB-factors for general width functions is described by the solution of a certain equation (see (3) below). Moreover, the results are of different nature depending on the rate of $w(t)$. We also introduce the notion of *inner cusps* and show that for them the rates of MB-factors improve substantially. (Fig. 3 below shows a "usual" cusp, while Fig. 2 illustrates an inner cusp.) Also, it turns out that the estimates improve further if the inner cusp has a *central-symmetric twist* (see Fig. 6). In order to verify the sharpness of our estimates of MB-factors, an auxiliary extremal problem for weighted polynomials is studied. (This extremal problem, which is interesting in itself, also leads to Eq. (3).) Finally, we would like to emphasize that the methods of this paper are completely elementary. We shall rely extensively on univariate Markov, Bernstein and Remez-type inequalities and their variations. In addition, a delicate construction of polynomial curves embedded into the domain will be required. Thus our exposition will be self-contained and will not use plurisubharmonic methods.

In order to introduce the class of cusps studied in this paper we shall need some additional notations. Set $\mathbf{F}^m := \{\mathbf{u} = (u_1, ..., u_m) \in \mathbf{R}^m : u_j = \pm 1, 1 \leqslant j \leqslant m\}$. Furthermore, denote by $\mathbf{F}^m(a, \mathbf{x}, \mathbf{u})$ the m-dimensional cube in the direction $\mathbf{u} \in \mathbf{F}^m$ with edge $a \in \mathbf{R}^+$ and vertex $\mathbf{x} = (x_1, ..., x_m) \in \mathbf{R}^m$, i.e.,

$$\mathbf{F}^m(a, \mathbf{x}, \mathbf{u}) := \{\mathbf{y} = (y_1, ..., y_m) \in \mathbf{R}^m : 0 \leqslant u_j(y_j - x_j) \leqslant a, \ 1 \leqslant j \leqslant m\}.$$

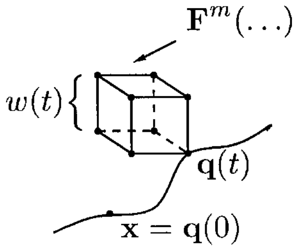

FIGURE 1

Finally, we consider a function $w \in C^1[0, 1]$ such that $0 = w(0) < w(t)$, $t \in (0, 1]$, and $w(t)/t$ is monotone increasing on $(0, 1]$.

Definition 1. We say that $\mathbf{x} \in \mathbf{K}$ is a w-cusp for the domain $\mathbf{K}$ if there exists a polynomial curve $\mathbf{q}\colon [-1, 1] \to \mathbf{R}^m$ and $\mathbf{u}, \mathbf{v} \in \mathbf{F}^m$ such that $\mathbf{q}(0) = \mathbf{x}$ and $\mathbf{F}(w(t), \mathbf{q}(t), \mathbf{u}) \subset \mathbf{K}$ for $0 < t \leqslant 1$, $\mathbf{F}(w(|t|), \mathbf{q}(t), \mathbf{v}) \subset \mathbf{K}$ for $-1 \leqslant t < 0$ (Fig. 1).

The size of the cubes $\mathbf{F}^m(\cdots)$ imbedded into $\mathbf{K}$ can rapidly shrink as $|t| \to 0$, i.e., around the point $\mathbf{x}$, in particular if $w(t) = o(t)$ this allows us to consider non Lip 1 points of bd $\mathbf{K}$ (i.e., *real* cusps). The function $w(t)$ is measuring the width of the cusp around $\mathbf{x}$. Typically, we may have a cusp at $\mathbf{x}$ with the curve $\mathbf{q}$ *passing through* $\mathbf{x}$ (see Fig. 2), or $\mathbf{x}$ may be the *endpoint* of $\mathbf{q}$ (Fig. 3). The fact that the curve $\mathbf{q}(t)$ in the above definition is parametrized on $[-1, 1]$ with $\mathbf{x} = \mathbf{q}(0)$ does not exclude from consideration *terminal cusps* of the type shown in Fig. 3. Indeed, given a polynomial curve $\mathbf{q}\colon [0, 1] \to \mathbf{R}^m$ with endpoint at $\mathbf{x} = \mathbf{q}(0)$ we can always consider another polynomial curve $\tilde{\mathbf{q}}(t) := \mathbf{q}(t^2)\colon [-1, 1] \to \mathbf{R}^m$ with $\mathbf{x} = \tilde{\mathbf{q}}(0)$ being its "inner" point. (Of course, such a reparametrization changes the width function $w(t)$ as well.) Thus both *inner* and *terminal* cusps (Fig. 2 and 3, resp.) are covered by the above definition. On the other hand, we shall see below that the magnitude of the MB factors is substantially smaller for inner cusps. Also, it should be noted that in the above definition of w-cusps

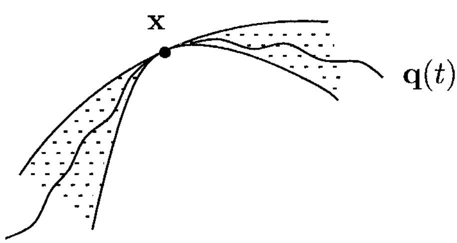

FIGURE 2

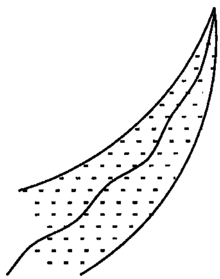

FIGURE 3

the cubes $\mathbf{F}^m(w(|t|), \mathbf{q}(t), \mathbf{u})$ imbedded into $\mathbf{K}$ have their *vertices* (and not centers) on $\mathbf{q}$, i.e., they are required only to "touch" $\mathbf{q}$ (and not necessarily "cover" it). This will allow us to consider more general domains when a cusp $\mathbf{x} \in \operatorname{bd} \mathbf{K}$ can be connected to $\mathbf{K}$ by a polynomial curve $\mathbf{q}$ contained in bd $\mathbf{K}$, but no polynomial curve connects $\mathbf{x}$ to int $\mathbf{K}$. Finally, let us note that the vectors $\mathbf{u}, \mathbf{v} \in \mathbf{F}^m$ specify the orientation of the embedded cubes in the above definition. Moreover, this orientation is fixed for $t > 0$ and $t < 0$, and may change (if $\mathbf{u} \neq \mathbf{v}$) as $\mathbf{q}$ passes through $\mathbf{x}$. Thus we allow the domain to "twist" around a cusp. The special case when $\mathbf{u} = -\mathbf{v}$ will be called a *central-symmetric twist*.

In the next section we shall present our main results on the size of the MB factors at cuspidal points. This will be followed by various examples and applications. Then proofs will be provided and, finally, we shall mention some open problems.

2. NEW RESULTS

In order to formulate our main theorems we shall need the quantity $\delta_n(w)$ defined as the unique solution of the equation

$$nt = \log \frac{1}{w(t)} \qquad \text{for} \quad n \geqslant \log \frac{1}{w(1)}. \tag{3}$$

Clearly, $\delta_n(w) \downarrow 0+$ and $n\delta_n(w) \to \infty$ as $n \to \infty$. Moreover, it is easy to see that $\delta_{cn} \sim \delta_n$ as $n \to \infty$. We shall use these properties without further references.

Let us also consider some typical w-cuspidal domains which will provide the needed lower bounds:

$$\mathbf{K}_1(w) := \{\mathbf{x} \in \mathbf{R}^m \colon |x_1| \leqslant w(\sqrt{x_j}),\ 0 \leqslant x_j \leqslant 1,\ 2 \leqslant j \leqslant m\} \tag{4}$$

and

$$\mathbf{K}_2(w) := \{\mathbf{x} \in \mathbf{R}^m \colon |x_1| \leqslant w(|x_j|),\ |x_j| \leqslant 1,\ 2 \leqslant j \leqslant m\}. \tag{5}$$

Then $\mathbf{K}_1(w)$ and $\mathbf{K}_2(w)$ have a terminal and inner w-cusp at $\mathbf{0}$, respectively.

THEOREM 1. *Let* $\mathbf{K} \subset \mathbf{R}^m$ *and* $\mathbf{x} \in \operatorname{bd} \mathbf{K}$ *be a* w*-cusp, where* $w(t)/t^2$ *is increasing on* $(0, 1]$. *Then for every* $n \in \mathbf{N}$

$$\log M_n(\mathbf{K}, \mathbf{x}) \leqslant cn\, \delta_n(w), \tag{6}$$

where c *depends only on the degree of the polynomial curve related to the cusp. On the other hand*

$$\log M_n(\mathbf{K}_i(w), \mathbf{0}) \geqslant c_1 n\, \delta_n(w), \qquad i = 1, 2; \quad n \in \mathbf{N}, \tag{7}$$

with an absolute constant $c_1 > 0$.

Theorem 1 provides sharp asymptotic bounds for $\log M_n(\mathbf{K}, \mathbf{x})$ when $\mathbf{x}$ is a w-cusp. In case when w has polynomial growth, i.e., for some $\beta > 1$ the function $w(t)/t^\beta$ is decreasing on $[0, 1]$, we can improve the above result and determine the magnitude of $M_n(\mathbf{K}, \mathbf{x})$ (and not its logarithm).

THEOREM 2. *Let* $\mathbf{K} \subset \mathbf{R}^m$ *and* $\mathbf{x} \in \operatorname{bd} \mathbf{K}$ *be a* w*-cusp, where* $w(t)/t^\beta$ *is increasing for* $\beta = 2$ *but decreasing for some* $\beta > 2$ *on* $(0, 1]$. *Then for every* $n \in \mathbf{N}$

$$M_n(\mathbf{K}, \mathbf{x}) \leqslant \frac{c}{w\left(\frac{1}{n}\right)}, \tag{8}$$

where c *depends only on the degree of the polynomial curve related to the cusp. Conversely*

$$M_n(\mathbf{K}_i(w), \mathbf{0}) \geqslant \frac{c_1}{w\left(\frac{1}{n}\right)}, \qquad i = 1, 2; \quad n \in \mathbf{N}, \tag{9}$$

with an absolute constant $c_1 > 0$.

Theorems 1 and 2 give rise to an extension of our results to global MB-factors:

DEFINITION 2. We shall say that $\mathbf{K}$ is uniformly w-cuspidal if every $\mathbf{x} \in \mathbf{K}$ is a w-cusp in the sense of Definition 1 where the degree of polynomial curves connecting $\mathbf{x}$ to the domain is *uniformly* bounded for all $\mathbf{x} \in \mathbf{K}$.

With this definition we easily obtain the following

COROLLARY 1. *Let $\mathbf{K} \subset \mathbf{R}^m$ be uniformly w-cuspidal. Then the upper estimates of Theorems 1–2 remain valid for the global Markov factors $M_n(\mathbf{K})$. Moreover, these estimates are sharp, in general.*

Theorems 1 and 2 cover the case when $w(t)/t^2$ is increasing. It remains now to consider w-cusps with $w(t)/t^2$ decreasing.

THEOREM 3. *Let $\mathbf{K} \subset \mathbf{R}^m$ and $\mathbf{x} \in \operatorname{bd} \mathbf{K}$ be a w-cusp, where $w(t)/t^2$ is decreasing on $(0, 1]$. Then for every $n \in \mathbf{N}$*

$$M_n(\mathbf{K}, \mathbf{x}) \leqslant cn^2. \tag{10}$$

Moreover, if $\mathbf{K}$ has a central-symmetric twist at $\mathbf{x}$ (i.e., $\mathbf{u} = -\mathbf{v}$) then

$$M_n(\mathbf{K}, \mathbf{x}) \leqslant \frac{c_1}{w\left(\dfrac{1}{n}\right)}. \tag{11}$$

Here the constants depend only on the degree of the polynomial curve related to the cusp.

The sharpness of estimates (10) and (11) will follow from a general lower bound for $M_n(\mathbf{K}, \mathbf{x})$ given below.

Consider a positive function $\varphi \in C[0, \infty)$ such that $\varphi(t)/t$ is increasing and $\varphi(t)/t^2$ is decreasing in $(0, \infty)$. Then the set

$$\mathbf{B}(\varphi, r) := \left\{ \mathbf{x} = (x_1, ..., x_m) \in \mathbf{R}^m \colon \sum_{j=1}^{m} \varphi(|x_j|) \leqslant \varphi(r) \right\}$$

is called a φ-ball in $\mathbf{R}^m$ of radius r. (When $\varphi(t) = t^p$, $p > 0$, these sets are the usual l_p-balls.) Moreover, the points $(\pm r, 0, ..., 0)$, ..., $(0, ..., 0, \pm r)$ are the *vertices* of $\mathbf{B}(\varphi, r)$. In addition, translations and rotations of $\mathbf{B}(\varphi, r)$ are also called φ-balls (with corresponding vertices and centers). Using this notion we can give a general lower bound for local MB factors which in particular implies the sharpness of the estimates (10) and (11).

THEOREM 4. *Let $\mathbf{K} \subset \mathbf{R}^m$ and $\mathbf{x} \in \operatorname{bd} \mathbf{K}$. Suppose there exists a φ-ball with vertex at $\mathbf{x}$ whose only common point with $\mathbf{K}$ is $\mathbf{x}$. Then for every $n \in \mathbf{N}$*

$$M_n(\mathbf{K}, \mathbf{x}) \geqslant \frac{c}{\varphi\left(\frac{1}{n}\right)}. \tag{12}$$

For a wide class of cusps, Theorem 4 is applicable with $\varphi(t) = t^2$ (l_2-ball) yielding that (10) is sharp, in general. Moreover, applying Theorem 4 for the set $\tilde{\mathbf{K}}_p$ of Example 1 below ($1 < p < 2$, Fig. 6) shows that (11) is sharp, in general, too.

Finally, let us note that if $\mathbf{x}$ is a terminal w-cusp (as shown in Fig. 3), the above results should be applied with the width function $\tilde{w}(t) := w(t^2)$. Since our basic assumption on w is that $w(t)/t$ is increasing, it follows that $\tilde{w}(t)/t^2$ is increasing as well, i.e., Theorem 3 is not applicable for terminal cusps. On the other hand, Theorems 1 and 2 will hold with w replaced by $\tilde{w}$.

3. EXAMPLES AND APPLICATIONS

For simplicity, all the examples below will be constructed in $\mathbf{R}^2$. First we shall look at various domains when the width function $w(t)$ is of polynomial growth.

EXAMPLE 1. For $p > 1$ set

$$\mathbf{K}_p := \{(x, y) \in \mathbf{R}^2 \colon 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant x^p\},$$
$$\mathbf{K}_p^* := \{(x, y) \in \mathbf{R}^2 \colon |x| \leqslant 1, 0 \leqslant y \leqslant |x|^p\},$$

and

$$\tilde{\mathbf{K}}_p := \{(x, y) \in \mathbf{R}^2 \colon -1 \leqslant x \leqslant 1, 0 \leqslant y \operatorname{sgn} x \leqslant |x|^p\}$$

(see Fig. 4, 5, and 6, respectively).

These domains have w-cusps at $\mathbf{0}$ ($w(t) = t^{2p}$ for $\mathbf{K}_p$ and $w(t) = t^p$ for $\mathbf{K}_p^*$ and $\tilde{\mathbf{K}}_p$), with the line segment $\{(t, 0)\}$ $t \in [0, 1]$ or $[-1, 1]$ being the connecting polynomial curve. Thus by (8) and (9) we recover Goetgheluck's estimate $M_n(\mathbf{K}_p, \mathbf{0}) \sim n^{2p}$. The same estimates yield $M_n(\mathbf{K}_p^*, \mathbf{0}) \sim n^p$ if $p > 2$, while (10) and (12) (with $\varphi(t) = t^2$) yield that $M_n(\mathbf{K}_p^*, \mathbf{0}) \sim n^2$ when $1 \leqslant p \leqslant 2$. Thus we can see that the MB factors for $\mathbf{K}_p^*$ (inner cusp) improve considerably compared to $\mathbf{K}_p$ (terminal cusp). Further improvement is exhibited by the domain $\tilde{\mathbf{K}}_p$ of Fig. 6. Here (using again (8) and (9)) $M_n(\tilde{\mathbf{K}}_p, \mathbf{0}) \sim M_n(\mathbf{K}_p^*, \mathbf{0}) \sim n^p$ ($p \geqslant 2$). On the other hand, since $\tilde{\mathbf{K}}_p$ has a

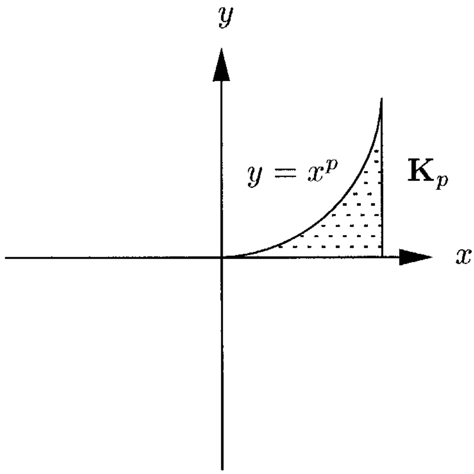

FIGURE 4

central-symmetric twist at the cusp $\mathbf{0}$, for $1 < p < 2$ we can use (11) and (12) (with $\varphi(t) = t^p$) yielding $M_n(\tilde{\mathbf{K}}_p, \mathbf{0}) \sim n^p$ in this case, as well. It is interesting to observe the somewhat surprising fact that the order of magnitude of the MB factor for $\tilde{\mathbf{K}}_p$ at $\mathbf{0}$ covers the complete range of powers n^p for $1 \leqslant p < \infty$.

The above considerations can be extended to the case when the cusp is connected to the set not necessarily by a line segment but some other polynomial curve.

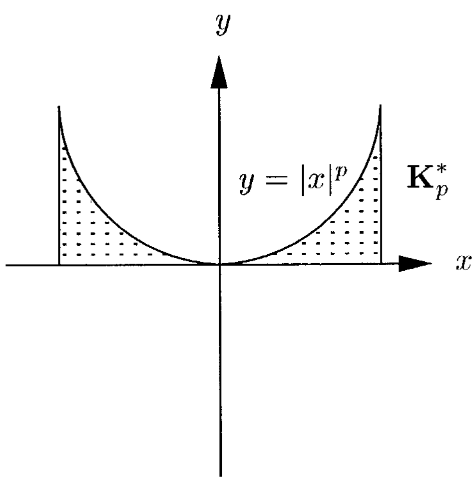

FIGURE 5

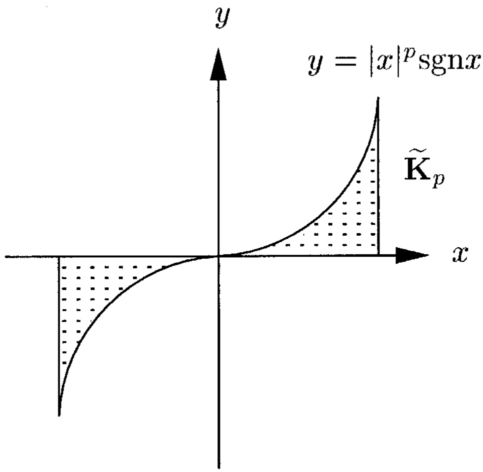

FIGURE 6

EXAMPLE 2. For $1 \leqslant \beta < \alpha$ set

$$\mathbf{K}_{\alpha,\beta} := \{(x, y) \in \mathbf{R}^2 : 0 \leqslant x \leqslant 1,\ x^\alpha \leqslant y \leqslant x^\beta\}$$

and

$$\mathbf{K}^*_{\alpha,\beta} := \{(x, y) \in \mathbf{R}^2 : |x| \leqslant 1,\ |x|^\alpha \leqslant y \leqslant |x|^\beta\}$$

(see Fig. 7 and 8). Then

$$M_n(\mathbf{K}_{\alpha,\beta}, \mathbf{0}) \begin{cases} \sim n^{2\beta} & \text{if} \quad 1 \leqslant \beta \leqslant [\alpha], \\ \leqslant c n^{2\beta(\alpha - [\alpha])/(\alpha-\beta)} & \text{if} \quad 1 \leqslant [\alpha] \leqslant \beta < \alpha \end{cases}$$

and

$$M_n(\mathbf{K}^*_{\alpha,\beta}, \mathbf{0}) \begin{cases} \sim n^{\beta} & \text{if} \quad 1 \leqslant \beta \leqslant [\alpha], \\ \leqslant c n^{\beta(\alpha - [\alpha])/(\alpha-\beta)} & \text{if} \quad 1 \leqslant [\alpha] \leqslant \beta < \alpha. \end{cases}$$

We shall prove only the first statement; the second can be proved similarly. In case $1 \leqslant \beta \leqslant [\alpha]$ we can use the polynomial curve $\mathbf{q}(t) = (t, t^{[\alpha]})$ in $\mathbf{K}_{\alpha,\beta}$, and a direct application of Theorem 2 yields the upper estimate. The sharpness of the estimate follows from the corresponding lower estimate for $\mathbf{K}_\beta$ of Example 1, since evidently $\mathbf{K}_{\alpha,\beta} \subset \mathbf{K}_\beta$.

In case $1 \leqslant [\alpha] \leqslant \beta < \alpha$ we consider the polynomial curve

$$\mathbf{q}_n(t) = (t, c_n t^{[\alpha]}) \qquad \text{with} \quad c_n := n^{-2((\alpha-[\alpha])(\beta-[\alpha]))/(\alpha-\beta)}$$

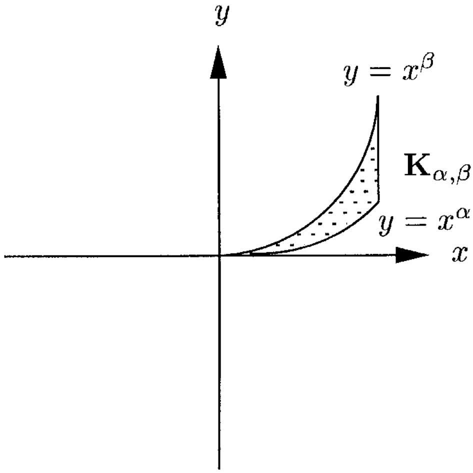

FIGURE 7

which lies in $\mathbf{K}_{\alpha,\beta}$ if $x_1 \leqslant t \leqslant x_2$, where

$$x_1 := n^{-2(\alpha - [\alpha])/(\alpha - \beta)} \qquad \text{and} \qquad x_2 := n^{-2(\beta - [\alpha])/(\alpha - \beta)}.$$

We wish to apply (8) in Theorem 2 for the boundary points $\mathbf{x}(t) := (t, c_n t^{[\alpha]})$ of the domain

$$\mathbf{K}_{n,\alpha,\beta} := \{(x, y) \in \mathbf{K}_{\alpha,\beta} : y \leqslant c_n x^{[\alpha]}\} \subset \mathbf{K}_{\alpha,\beta}.$$

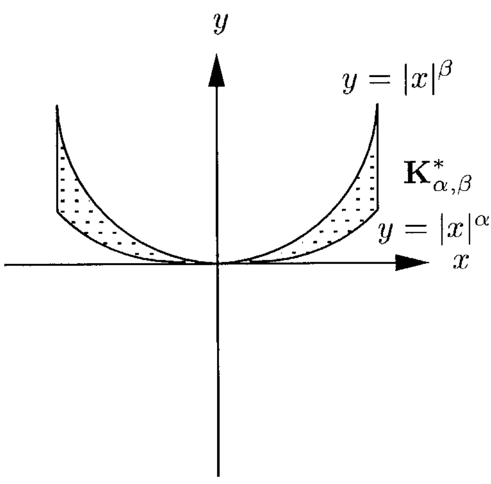

FIGURE 8

Elementary calculations yield that $w(t) \sim x_1^{\beta-1} x_2 t^2$. (It should be noted that the domain $\mathbf{K}_{n,\alpha,\beta}$ depends on n, but the degree of the connecting curve $\mathbf{x}(t)$ is independent of n. Thus in view of the Remark following Theorem 3 we can apply estimate (8) in these circumstances.) Hence

$$M_n(\mathbf{K}_{\alpha,\beta}, \mathbf{x}(t)) \leqslant M_n(\mathbf{K}_{n,\alpha,\beta}, \mathbf{x}(t)) \leqslant c \frac{n^2}{x_1^{\beta-1} x_2} = cn^{2\beta(\alpha-[\alpha])/(\alpha-\beta)},$$

$$x_1 \leqslant t \leqslant x_2.$$

The latter is nothing but an estimate for the partial derivatives on the curve $\mathbf{x}(t)$, $t \in [x_1, x_2]$, which are polynomials of t of degree at most $n[\alpha]-1$. Since $x_1/x_2 = 1/n^2$, this estimate extends to $\mathbf{0}$ by Remez inequality (see (15)), and this proves our statement.

Now we shall present some applications of Theorem 1 which are concerned with w-cusps whose width function w decreases to 0 faster than any power of t.

EXAMPLE 3. For $\alpha > 0$, set

$$\mathbf{D} := \{(x, y) \in \mathbf{R}^2 : 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant e^{-1/x^\alpha}\}$$

and

$$\mathbf{D}^* := \{(x, y) \in \mathbf{R}^2 : |x| \leqslant 1, |y| \leqslant e^{-1/|x|^\alpha}\}.$$

Then $\mathbf{0}$ is a w-cusp for $\mathbf{D}$ and $\mathbf{D}^*$ with $w(t) = e^{-1/t^{2\alpha}}$ and $w(t) = e^{-1/|t|^\alpha}$, respectively. It should be noted that the line segment $\{(t, 0)\} : t \in [0, 1]$ or

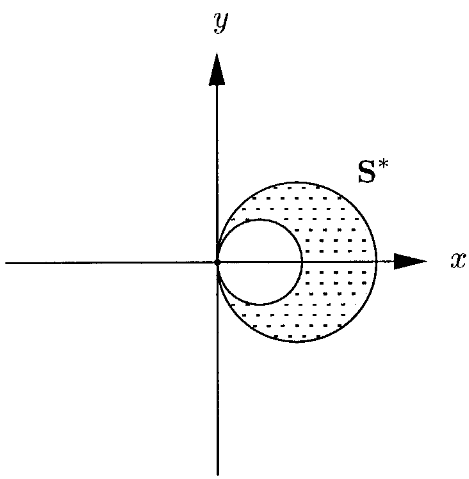

FIGURE 9

$[-1, 1]$ provides the only polynomial curve connecting $\mathbf{0}$ to the domains. Then by Theorem 1 we have

$$\log M_n(\mathbf{D}, \mathbf{0}) \sim n^{2\alpha/(1+2\alpha)} \qquad \text{and} \qquad \log M_n(\mathbf{D}^*, \mathbf{0}) \sim n^{\alpha/(1+\alpha)}.$$

EXAMPLE 4. Let $\mathbf{S}^* \subset \mathbf{R}^2$ be the domain inside the circle $(x-2)^2 + y^2 = 4$ and outside the circle $(x-1)^2 + y^2 = 1$ (see Fig. 9). Then $\mathbf{0}$ is a $w(t) = ct^2$-cusp for $\mathbf{S}^*$ with the polynomial curve $\mathbf{q}(t) := \{(t^2/3, t): |t| \leqslant 1\} \subset \mathbf{S}^*$ passing through $\mathbf{0}$. Thus by Theorem 2 $M_n(\mathbf{S}^*, \mathbf{0}) \leqslant c_1 n^2$. Moreover, applying Theorem 4 (with $w(t) = t^2$) yields $M_n(\mathbf{S}^*, \mathbf{0}) \geqslant c_2 n^2$.

4. AUXILIARY RESULTS

In order to verify our main results we shall need some auxiliary statements. First we shall address the following extremal problem for weighted polynomials. Let $w \in C[0, 1]$ be an increasing function with $w(0) = 0$. Then given $p \in \mathscr{P}_n^1$ such that $w(\sqrt{x})\, |p(x)| \leqslant 1$ for $x \in [0, 1]$, or $w(|x|)\, |p(x)| \leqslant 1$ for $x \in [-1, 1]$, how large can $|p(0)|$ be? Thus we consider the problem of determining the order of magnitude of

$$\Delta_n^{(1)}(w) := \max\{|p(0)| : \|w(\sqrt{x})\, p(x)\|_{C[0,1]} \leqslant 1, p \in \mathscr{P}_n^1\}$$

and

$$\Delta_n^{(2)}(w) := \max\{|p(0)| : \|w(|x|)\, p(x)\|_{C[-1,1]} \leqslant 1, p \in \mathscr{P}_n^1\}.$$

PROPOSITION 1. *If $\delta_n(w)$ is the solution of the Eq.* (3), *then*

$$c_2\, \delta_n(w) \leqslant \log \Delta_n^{(i)}(w) \leqslant c_1\, \delta_n(w), \qquad i = 1, 2; \quad n \in \mathbf{N}. \tag{13}$$

Moreover, if w is of polynomial growth then

$$\frac{c_4}{w\left(\dfrac{1}{n}\right)} \leqslant \Delta_n^{(i)}(w) \leqslant \frac{c_3}{w\left(\dfrac{1}{n}\right)}, \qquad i = 1, 2; \quad n \in \mathbf{N}. \tag{14}$$

It turns out that the lower bounds in (13) and (14) easily imply (7) and (9). First we show how this can be accomplished, and then the proof of Proposition 1 will be given.

Proof of the Lower Bounds in Theorems 1 *and* 2. Let $p_i \in \mathscr{P}_n^1$ be the extremal polynomial for $\Delta_n^{(i)}(w)$, $i = 1, 2$ (the existence of such a p_i is obvious). Set $p_i^*(\mathbf{x}) := x_1 p_i(x_2)$, $i=1, 2$, where $\mathbf{x}=(x_1, ..., x_m) \in \mathbf{R}^m$. Then $p_i^* \in \mathcal{P}_{n+1}^m$, and for $\mathbf{x} \in \mathbf{K}_i(w)$ we have

$$|p_i^*(\mathbf{x})| \leqslant w(|x_2|^{i/2})\,|p_i(x_2)| \leqslant 1, \qquad i=1, 2.$$

On the other hand,

$$\left|\frac{\partial}{\partial x_1}\, p_i^*(\mathbf{0})\right| = |p_i(0)| = \varDelta_n^{(i)}(w), \qquad i=1, 2.$$

Thus $M_n(\mathbf{K}_i(w), \mathbf{0}) \geqslant \varDelta_n^{(i)}(w)$, $i=1, 2$; $n \in \mathbf{N}$. Hence (7) and (9) follow immediately from (13) and (14). ▮

The following Remez-type inequalities play a central role in our considerations.

Lemma 1. *Let* $p_n \in \mathcal{P}_n^1$ *be such that* $\operatorname{mes}\{x \in [0, 1] : |p_n(x)| > 1\} \leqslant h$ *with some* $0 < h \leqslant 1/2$. *Then*

$$\|p_n\|_{C[0,1]} \leqslant e^{c_1 n \sqrt{h}}. \tag{15}$$

Moreover, if $|p_n(x)| \leqslant 1$ *for* $h \leqslant |x| \leqslant 1$ *then*

$$\|p_n\|_{C[-1,1]} \leqslant e^{c_2 n h}. \tag{16}$$

Here $c_1, c_2 > 0$ *are some absolute constants.*

Inequality (15) is the standard Remez inequality (see [2], p. 227). The sharper estimate (16) follows by the substitution $x = \cos t$ from the trigonometric version of the Remez inequality given in [3]. We need a similar sharper version of Markov's inequality which is provided by the next statement.

Lemma 2. *Let* $p_n \in \mathcal{P}_n^1$ *satisfy* $|p_n(x)| \leqslant 1$ *for* $h \leqslant |x| \leqslant 1$, *where* $0 < h \leqslant \frac{1}{2}$. *Then*

$$|p_n'(\pm h)| \leqslant c(n + hn^2).$$

Proof. Since $p_n = p_n^* + \tilde{p}_n$, where p_n^* and $\tilde{p}_n$ are even and odd polynomials, respectively, bounded by 1 for $h \leqslant |x| \leqslant 1$, it suffices to consider the cases when p_n is even or odd.

Case 1. p_n is even. Then $p_n(x) = g(x^2)$ with $g \in \mathcal{P}_r^1$ $(r \leqslant [n/2])$, where $|g(y)| \leqslant 1$ for $h^2 \leqslant y \leqslant 1$. Hence by (1) $|g'(h^2)| \leqslant 2r^2/(1-h^2)$. Thus

$$|p_n'(\pm h)| = 2h\,|g'(h^2)| \leqslant \tfrac{4}{3} h n^2.$$

Case 2. p_n is odd. Then $p_n(x)=xg(x^2)$, where $g\in\mathscr{P}_r^1$ $(r\leqslant[(n-1)/2])$ and $|g(y)|\leqslant 1/h$ for $h^2\leqslant y\leqslant 1$. If $h\leqslant 1/n$ then by (16) $\|p_n\|_{C[-1,1]}\leqslant c_0$ and $|p_n'(\pm h)|\leqslant n/\sqrt{1-h^2}\leqslant cn$ by (2). Thus we may assume that $h>1/n$, i.e., $|g(y)|\leqslant n$ for $h^2\leqslant y\leqslant 1$. Hence using again (1)

$$|p_n'(\pm h)|\leqslant|g(h^2)|+2h^2\,|g'(h^2)|\leqslant n+2h^2\,\frac{2r^2\,\dfrac{1}{h}}{1-h^2}\leqslant n+chn^2.$$

This completes the proof of Lemma 1. ∎

Proof of Proposition 1. The upper bounds in (13) and (14) follow directly from Lemma 1. Indeed, if $w(\sqrt{x})\,|p_n(x)|\leqslant 1$ for $x\in[0,1]$ then $|p_n(x)|\leqslant 1/w(\sqrt{h})$ if $h\leqslant x\leqslant 1$ $(h>0)$. Hence by (15)

$$|p_n(x)|\leqslant\frac{e^{c_1 n\sqrt{h}}}{w(\sqrt{h})},\qquad 0\leqslant x\leqslant 1,\tag{17}$$

where $h>0$ can be chosen arbitrarily. Setting now $h:=1/n^2$ in (17) yields the upper bound of (14) for $i=1$. Moreover, choosing $h:=\delta_n(w)^2$ implies the upper bound of (13) for $i=1$. The upper estimates for $\varDelta_n^{(2)}(w)$ follow analogously from (16).

The proof of the lower bounds of Proposition 1 is somewhat more involved. We shall verify the lower bound for $\varDelta_n^{(1)}(w)$ by choosing the extremal polynomial as

$$p_n^*(x):=\frac{T_{n+k}^{(k)}(x)}{T_{n+k}^{(k)}(0)}$$

where $T_n(x)=\cos n\arccos(x-1)$ $(n\in\mathbf{N},\ x\in[0,2])$ is the Chebyshev polynomial on $[0,2]$, and the positive integer $k\leqslant n$ will be determined below. We shall need two auxiliary estimates:

$$|T_n^{(m)}(0)|\geqslant\frac{(n-m)^{2m}}{m^m},\qquad m\leqslant n,\tag{18}$$

and

$$|T_n^{(m)}(x)|\leqslant\left(\frac{2n}{\sqrt{x(2-x)}}\right)^m,\qquad x\in(0,2)\tag{19}$$

(see [2], pp. 256 and 258, respectively).

Clearly, $|p_n^*(0)|=1$. In order to obtain a lower bound for $\Delta_n^{(1)}(w)$, it suffices now to estimate $w(\sqrt{x})\,|p_n^*(x)|$ on $[0,1]$ from above. Evidently, $|p_n^*(x)|\leqslant 1$ for $x\in[0,1]$, i.e.,

$$w(\sqrt{x})\,|p_n^*(x)|\leqslant w(\sqrt{x}),\qquad x\in[0,1]. \tag{20}$$

Moreover, by (19) and (18)

$$w(\sqrt{x})\,|p_n^*(x)|\leqslant\left(\frac{4n}{\sqrt{x(2-x)}}\right)^k\cdot\frac{k^k}{n^{2k}}\,w(\sqrt{x})\leqslant\left(\frac{4k}{n}\right)^k\cdot\frac{w(\sqrt{x})}{x^{k/2}},\qquad 0<x\leqslant 1. \tag{21}$$

Assume first that $x^{-\beta}w(x)$ is decreasing for some $\beta>0$. In this case we set $k:=[\beta]+1>\beta$. Using (21) for $k^2/n^2\leqslant x\leqslant 1$ yields

$$w(\sqrt{x})\,|p_n^*(x)|\leqslant 4^k w\left(\frac{k}{n}\right).$$

If $0\leqslant x\leqslant k^2/n^2$ then (20) implies directly that $w(\sqrt{x})\,|p_n^*(x)|\leqslant w(k/n)$. Combining the last two estimates we obtain that $w(\sqrt{x})\,|p_n^*(x)|\leqslant cw(1/n)$ for $0\leqslant x\leqslant 1$. This verifies the lower bound for $\Delta_n^{(1)}(w)$ in (14). In general, we can set $k:=[n\,\delta_n(w)/(4e)]$. Then using again (20) for $0\leqslant x\leqslant(4ek/n)^2$ we have

$$w(\sqrt{x})\,|p_n^*(x)|\leqslant w(4ek/n)\leqslant w(\delta_n(w))=e^{-n\delta_n(w)}.$$

On the other hand, if $(4ek/n)^2\leqslant x\leqslant 1$ it follows by (21) that

$$w(\sqrt{x})\,|p_n^*(x)|\leqslant\left(\frac{4k}{n}\right)^k\left(\frac{n}{4ek}\right)^k w(1)=w(1)\,e^{-k}\leqslant e^{-cn\,\delta_n(w)}.$$

Combining again the last two estimates yields the lower bound in (13) for $i=1$. The lower bounds for $\Delta_n^{(2)}(w)$ follow from those of $\Delta_n^{(1)}(w)$ by the standard substitution $x=t^2$. ∎

Proof of the Upper Bounds in Theorems 1–3. Assume first that $w(t)/t^2$ is increasing in $[0,1]$, which is the assumption of Theorems 1 and 2. Let $\mathbf{x}\in\operatorname{bd}\mathbf{K}$ be a w-cusp. Then by the definition of w-cuspidal points there exists a polynomial curve $\mathbf{q}\colon[-1,1]\to\mathbf{R}^m$ and $\mathbf{u},\mathbf{v}\in\mathbf{F}^m$ such that $\mathbf{q}(0)=\mathbf{x}$ and $\mathbf{F}(w(t),\mathbf{q}(t),\mathbf{u})\subset\mathbf{K}$ $(0<t\leqslant 1)$, $\mathbf{F}(w(t),\mathbf{q}(t),\mathbf{v})\subset\mathbf{K}$ $(-1\leqslant t<0)$. Let

$\mathbf{u}=(u_1, ..., u_m)$, $\mathbf{v}=(v_1, ..., v_m)$ $(u_j, v_j=\pm 1)$ and $\mathbf{e}_k := \{\delta_{jk}\}_{j=1}^m = (0, ..., 0, 1, 0, ..., 0)$ $(1 \leqslant k \leqslant m)$. For arbitrary $0 < h \leqslant 1/2$ and $1 \leqslant k \leqslant m$ consider the polynomial curves

$$\mathbf{q}_h^{[k]}(t) := \mathbf{q}(t) + \begin{cases} \mathbf{e}_k u_k \dfrac{w(h)}{h^2}(t^2-h^2), & \text{if} \quad u_k = v_k, \\ \mathbf{e}_k u_k t \dfrac{w(h)}{h}, & \text{if} \quad u_k = -v_k \end{cases} \qquad (k=1, ..., m). \tag{22}$$

Since $w(t)/t^2$ is increasing we have

$$0 \leqslant \frac{w(h)}{h^2}(t^2-h^2) \leqslant \frac{w(|t|)}{t^2}(t^2-h^2) \leqslant w(|t|), \qquad h \leqslant |t| \leqslant 1. \tag{23}$$

Moreover, using also the (weaker) assumption that $w(t)/t$ increases yields that

$$0 \leqslant |t| \frac{w(h)}{h} \leqslant w(|t|), \qquad h \leqslant |t| \leqslant 1. \tag{24}$$

Let us denote by $l_{h,j}^{[k]}(t)$ and $l_j(t)$ the jth components of the curves $\mathbf{q}_h^{[k]}$ and $\mathbf{q}$, respectively. Evidently, $l_{h,j}^{[k]} \equiv l_j$ if $j \neq k$. Moreover, we have by (22)

$$u_k(l_{h,k}^{[k]}(t) - l_k(t)) = \begin{cases} \dfrac{w(h)}{h^2}(t^2-h^2), & \text{if} \quad u_k = v_k, \\ t\dfrac{w(h)}{h}, & \text{if} \quad u_k = -v_k \end{cases}$$

which implies

$$v_k(l_{h,k}^{[k]}(t) - l_k(t)) = \begin{cases} \dfrac{w(h)}{h^2}(t^2-h^2), & \text{if} \quad u_k = v_k, \\ -\, t\dfrac{w(h)}{h}, & \text{if} \quad u_k = -v_k. \end{cases}$$

Hence and by (23) and (24)

$$\begin{aligned} \mathbf{q}_h^{[k]}(t) &\subset \mathbf{F}(w(t), \mathbf{q}(t), \mathbf{u}) \subset \mathbf{K}, && \text{if} \quad h \leqslant t \leqslant 1, \\ \mathbf{q}_h^{[k]}(t) &\subset \mathbf{F}(w(t), \mathbf{q}(t), \mathbf{v}) \subset \mathbf{K}, && \text{if} \quad -1 \leqslant t \leqslant -h. \end{aligned} \tag{25}$$

Consider now the polynomial

$$g_k(t) := p_n(\mathbf{q}_h^{[k]}(t)), \qquad 0 \leqslant k \leqslant m; \quad \mathbf{q}_h^{[0]} := \mathbf{q},$$

where $p_n \in \mathscr{P}_n^m$ is an arbitrary polynomial such that $\|p_n\|_{C(\mathbf{K})} \leqslant 1$. Then $\deg g_k \leqslant cn$, and by (25)

$$|g_k(t)| \leqslant 1, \qquad h \leqslant |t| \leqslant 1; \quad 0 \leqslant k \leqslant m. \tag{26}$$

Let first $1 \leqslant k \leqslant m$ be such that $u_k = -v_k$. Note that in this case $\mathbf{q}_h^{[k]}(0) = \mathbf{q}(0) = \mathbf{x}$. Hence and by (22)

$$g_0'(0) = \sum_{j=1}^{m} \frac{\partial p_n}{\partial x_j}(\mathbf{x}) \frac{\partial l_j}{\partial t}(0) \tag{27}$$

and

$$g_k'(0) = g_0'(0) + u_k \frac{w(h)}{h} \cdot \frac{\partial p_n}{\partial x_k}(\mathbf{x}). \tag{28}$$

Using (26) and (16) yields $\|g_k\|_{C[-1,1]} \leqslant e^{cnh}$. Thus by the Bernstein inequality (2), $|g_k'(0)| \leqslant ne^{cnh}$. Applying these estimates together with (28) implies the bound

$$\left|\frac{\partial p_n}{\partial x_k}(\mathbf{x})\right| \leqslant \frac{2hn}{w(h)} e^{cnh}. \tag{29}$$

Let now $1 \leqslant k \leqslant m$ be such that $u_k = v_k$. In this case $\mathbf{q}_h^{[k]}(\pm h) = \mathbf{q}(\pm h)$, and similarly to (27)–(28) we have by (22)

$$\begin{aligned} g_0'(\pm h) &= \sum_{j=1}^{m} \frac{\partial p_n}{\partial x_j}(\mathbf{q}(\pm h)) \frac{\partial l_j}{\partial t}(\pm h), \\ g_k'(\pm h) &= g_0'(\pm h) \pm 2u_k \frac{w(h)}{h} \cdot \frac{\partial p_n}{\partial x_k}(\mathbf{q}(\pm h)). \end{aligned} \tag{30}$$

Furthermore, using (26) and Lemma 2

$$|g_k'(\pm h)| \leqslant c(n + hn^2), \qquad 0 \leqslant k \leqslant m.$$

Applying these estimates in (30) yields that for every $0 \leqslant h \leqslant 1/2$

$$\left|\frac{\partial p_n}{\partial x_k}(\mathbf{q}(\pm h))\right| \leqslant c\,\frac{hn + h^2 n^2}{w(h)}.$$

Then using that both $w(t)/t$ and $w(t)/t^2$ are increasing we have

$$\left|\frac{\partial p_n}{\partial x_k}(\mathbf{q}(t))\right| \leqslant c\,\frac{hn + h^2 n^2}{w(h)}, \qquad h \leqslant |t| \leqslant 1/2.$$

Therefore applying again (16) for the univariate polynomial $\partial p_n/\partial x_k(\mathbf{q}(t))$ we have

$$\left|\frac{\partial p_n}{\partial x_k}(\mathbf{x})\right| = \left|\frac{\partial p_n}{\partial x_k}(\mathbf{q}(0))\right| \leqslant \frac{c(hn+h^2n^2)\, e^{c_1 nh}}{w(h)}, \qquad 0<h\leqslant 1/4. \tag{31}$$

Note that (31) is more general than (29) (obtained when $u_k=-v_k$), i.e., (31) holds for every $1\leqslant k\leqslant m$ and $0<h\leqslant 1/4$. Setting now $h:=\delta_n(w)$ in (31) yields

$$\left|\frac{\partial p_n}{\partial x_k}(\mathbf{x})\right| \leqslant c(\delta_n(w)\, n+\delta_n(w)^2\, n^2)\, e^{(c_1+1)\, n\, \delta_n(w)} \leqslant e^{c_2 n\, \delta_n(w)}, \qquad 1\leqslant k\leqslant m,$$

which is the upper bound of Theorem 1. Moreover, choosing $h:=1/n$ in (31) gives the upper bound of Theorem 2.

It remains now to verify the upper bounds of Theorem 3. First it should be noted that if the domain has a central-symmetric twist at the cusp $\mathbf{x}$, i.e., $\mathbf{u}=-\mathbf{v}$ then $u_k=-v_k$ for *every* $1\leqslant k\leqslant m$ and (22) reduces to

$$\mathbf{q}_h^{[k]}(t)=\mathbf{q}(t)+\mathbf{e}_k u_k t\,\frac{w(h)}{h}.$$

Recall that in this case we had the sharper estimate (29) which was obtained using that $w(t)/t$ is increasing (the assumption that $w(t)/t^2$ is increasing was not required). Thus setting $h=1/n$ in (29) yields (11). In general, when $w(t)/t^2$ decreases on $[0,1]$ we can replace $\mathbf{q}_h^{[k]}(t)$ of (22) by

$$\mathbf{q}_h^{[k]}(t):=\mathbf{q}(t)+\begin{cases}\mathbf{e}_k u_k w(1)(t^2-h^2), & \text{if} \quad u_k=v_k,\\ \mathbf{e}_k u_k t\,\dfrac{w(h)}{h}, & \text{if} \quad u_k=-v_k\end{cases} \qquad (1\leqslant k\leqslant m).$$

The case when $u_k=-v_k$ leads to (29) as above (with only the increase of $w(t)/t$ required). If $u_k=v_k$, then using that $w(t)/t^2$ decreases we have

$$0\leqslant w(1)(t^2-h^2)\leqslant w(1)\, t^2\leqslant w(|t|), \qquad h\leqslant |t|\leqslant 1.$$

Thus repeating the above considerations we can replace (31) by

$$\left|\frac{\partial p_n}{\partial x_k}(\mathbf{x})\right| \leqslant c\left(\frac{n}{h}+n^2\right) e^{c_1 nh}, \tag{32}$$

which is more general than (29) ($w(h)\geqslant ch^2$). Setting $h:=1/n$ in (32) we obtain now (10). This completes the proof of Theorems 1–3. ∎

Proof of Theorem 4. Without loss of generality we may assume that $\mathbf{x}=\mathbf{0}$ and $\mathbf{B}_r \cap \mathbf{K}=\{\mathbf{0}\}$, where

$$\mathbf{B}_r := \left\{ \mathbf{x}=(x_1, ..., x_m) \in \mathbf{R}^m \colon \varphi(|x_1-r|)+\sum_{k=2}^{m} \varphi(|x_k|) \leqslant \varphi(r) \right\}.$$

First we prove that

$$-\frac{m-1}{n^2} \leqslant \sum_{k=1}^{m} x_k^2 - \frac{\varphi(r)}{rn^2 \varphi\left(\frac{1}{n}\right)} x_1 \leqslant a,$$

$$a := \sup \left\{ \sum_{k=1}^{m} x_k^2 + r\,|x_1| \colon (x_1, ..., x_m) \in \mathbf{K} \right\} < \infty, \tag{33}$$

whenever $(x_1, ..., x_m) \in \mathbf{K}$ and $n \geqslant 1/r$.

The upper estimate in (33) follows directly from the fact that $\varphi(t)/t^2$ is monotone decreasing and from the definition of a. Now the lower estimate is trivial if $x_1 \leqslant 0$. The same is valid if $x_1 > r$, since in this case using again that $\varphi(t)/t^2$ decreases

$$x_1^2 - \frac{\varphi(r)}{rn^2 \varphi\left(\frac{1}{n}\right)} x_1 \geqslant x_1(x_1-r) \geqslant 0, \qquad n \geqslant \frac{1}{r}.$$

Thus it remains to handle the case $0 < x_1 < r$. By the condition $\mathbf{B}_r \cap \mathbf{K}=\{\mathbf{0}\}$ we have

$$\varphi(r-x_1)+\sum_{k=2}^{m} \varphi(|x_k|) \geqslant \varphi(r), \qquad (x_1, ..., x_m) \in \mathbf{K}. \tag{34}$$

Here, using that $\varphi(t)/t$ is increasing, we get

$$\varphi(r-x_1) \leqslant \frac{r-x_1}{r} \varphi(r) = \varphi(r) - \frac{\varphi(r)}{r} x_1, \qquad 0 < x_1 < r,$$

and thus by (34)

$$\frac{\varphi(r)}{r} x_1 \leqslant \sum_{k=2}^{m} \varphi(|x_k|), \qquad 0 < x_1 < r, \quad (x_1, ..., x_m) \in \mathbf{K}.$$

Hence

$$\sum_{k=1}^{m} x_k^2 - \frac{\varphi(r)}{rn^2\varphi\left(\frac{1}{n}\right)}\, x_1 \geqslant \sum_{k=2}^{m} \left(x_k^2 - \frac{\varphi(|x_k|)}{n^2\varphi\left(\frac{1}{n}\right)} \right),$$

$$0 < x_1 < r, \qquad (x_1, ..., x_m) \in \mathbf{K}.$$

Now if $|x_k| \leqslant 1/n$ then since $\varphi(t)$ is monotone increasing, the corresponding terms in the above sum are $\geqslant -1/n^2$. On the other hand, if $|x_k| \geqslant 1/n$ then by $\varphi(t)/t^2$ being monotone decreasing these terms are nonnegative. This completes the proof of (33).

After these preliminaries, let $T_n(x)$ be the Chebyshev polynomial of degree n with respect to the interval $[0, a]$ normalized by $T_n(0) = 1$, and consider the polynomial

$$p_n(\mathbf{x}) := T_n \left(\sum_{k=1}^{m} x_k^2 - \frac{\varphi(r)}{rn^2\varphi\left(\frac{1}{n}\right)}\, x_1 \right), \qquad \mathbf{x} = (x_1, ..., x_m)$$

of total degree $2n$. By (33) we have

$$\|p_n\|_{C(\mathbf{K})} \leqslant T_n\left(-\frac{m-1}{n^2}\right) \leqslant e^{cn\sqrt{(m-1)/n^2}} = O(1).$$

On the other hand,

$$\left| \frac{\partial p_n}{\partial x_1}(\mathbf{0}) \right| = \frac{\varphi(r)}{rn^2\varphi\left(\frac{1}{n}\right)}\, T_n'(0) = \frac{2\varphi(r)}{ar\varphi\left(\frac{1}{n}\right)}. \quad \blacksquare$$

5. ON THE SUBEXPONENTIAL GROWTH OF MB-FACTORS

Let $\mathbf{K} \subset \mathbf{R}^m$ be such that $\operatorname{int} \mathbf{K} \neq \varnothing$. It is well-known that whenever $p_n(x) = a_0 + a_1 x + \cdots + a_n x^n$ is such that $|p_n(x)| \leqslant 1$ for $0 \leqslant x \leqslant 1$ then $\max_{0 \leqslant j \leqslant n} |a_j| \leqslant c^n$ with some absolute constant $c > 0$. Since $\operatorname{int} \mathbf{K} \neq \varnothing$ it follows that $\mathbf{K}$ contains an m-dimensional cube. Thus combining the above univariate fact with a standard product-type argument yields that for any $p_n(\mathbf{x}) = \sum_{|\mathbf{j}| \leqslant n} a_{\mathbf{j}} \mathbf{x}^{\mathbf{j}}$ with $|p_n| \leqslant 1$ on $\mathbf{K}$ we have the similar bound $\max_{|\mathbf{j}| \leqslant n} |a_{\mathbf{j}}| \leqslant c^n$ (here $c > 0$ depends on $\mathbf{K}$). In particular, this implies that $\sup_{n \in \mathbf{N}} M_n(\mathbf{K})^{1/n} < \infty$ whenever $\operatorname{int} \mathbf{K} \neq \varnothing$. On the other hand, by estimate

(6) of Theorem 1 for any polynomial cusp $\mathbf{x} \in \mathbf{K}$ we have the subexponential growth

$$\limsup_{n \to \infty} M_n(\mathbf{K}, \mathbf{x})^{1/n} = 1 \tag{35}$$

(no matter how fast $w(t)$ decreases to 0 as $t \to 0+$). The following corollary to Theorem 1 shows that *any* subexponential growth can be achieved for a proper w-cusp.

COROLLARY 2. *Given an arbitrary sequence* $\{\varepsilon_n\}_{n=1}^{\infty} \downarrow 0$, $\varepsilon_1 = 1$, *there exist sets* $\mathbf{K}$ *with a polynomial cusp* $\mathbf{x} \in \mathbf{K}$ *such that*

$$\log M_n(\mathbf{K}, \mathbf{x}) \geqslant c_1 n \varepsilon_n, \qquad n = 1, 2, \ldots.$$

Proof. We define a width function w in the following way: let $w(\varepsilon_1) = 1$ and

$$w(\varepsilon_n) = \min\left\{\frac{\varepsilon_n^2}{\varepsilon_{n-1}^2}\, w(\varepsilon_{n-1}),\ e^{-n\varepsilon_n}\right\}, \qquad n = 2, 3, \ldots. \tag{36}$$

Furthermore, in each interval $[\varepsilon_n, \varepsilon_{n-1}]$, $n = 2, 3, \ldots$, let the function $w(t)/t^2$ be linear. Then evidently $w(t)/t^2$, and all the more $w(t)$ will be monotone increasing functions in $[0, 1]$. By (36), for the solution of the Eq. (3) we have $\delta_n(w) \geqslant \varepsilon_n$, $n = 1, 2, \ldots$. Thus for the domains $\mathbf{K} = \mathbf{K}_i(w)$, $i = 1, 2$, defined in (4)–(5) we get from (7)

$$\log M_n(\mathbf{K}_i(w), \mathbf{x}) \geqslant c_1 n\, \delta_n(w) \geqslant c_1 n \varepsilon_n, \qquad i = 1, 2; \quad n = 1, 2, \ldots,$$

which proves the corollary. ∎

Let us mention that in the univariate case Totik [8] verified the subexponential growth of MB-factors on regular Cantor-type sets and showed that any such growth can be achieved on them. Totik's result is an elegant contribution, but univariate Cantor sets have empty interior and thus they cannot be used for construction of "fat" sets in $\mathbf{R}^m$, $m \geqslant 2$. (The set $\mathbf{K} \in \mathbf{R}^m$ is called fat if $\overline{\operatorname{int}\ \mathbf{K}} = \mathbf{K}$.)

The following question arises naturally: when does (35) hold in general? We now provide some evidence showing that further conditions need to be imposed in order for (35) to hold at every $\mathbf{x} \in \overline{\operatorname{int}\ \mathbf{K}}$.

EXAMPLE 5. There exists a fat set $\mathbf{K}$ with connected interior and $\mathbf{x} \in \mathbf{K}$ for which (35) fails to hold.

In order to verify this claim we need to use a polynomial constructed in a recent paper [4]. In [4] a sequence of intervals $I = \bigcup_{j=1}^{\infty} [a_j, b_j]$ such that $0<b_{j+1}<a_j<b_j$, $j=1, 2, ...$; $\lim_{j\to\infty} a_j=0$ and polynomials $p_n\in\mathscr{P}_n^1$ $(n\in\mathbf{N})$ such that $\|p_n\|_{C(I)}\leqslant 1$ and $|p_n'(0)|\geqslant r^n$ $(n\in\mathbf{N}, r>1)$, are constructed. It follows that $\|p_n\|_{C[0,1]}\leqslant c^n$ with some $c>1$. Now set $q_n(x, y)=(1-y)^n p_n(x)\in\mathscr{P}_{2n}^2$ and

$$\mathbf{K}=\{(x, y): 0\leqslant x, y\leqslant 1 \text{ and either } x\in I \text{ or } 1-1/c\leqslant y\leqslant 1\}.$$

Then we easily obtain that $\|q_n\|_{C(\mathbf{K})}\leqslant 1$. On the other hand $Dq_n(\mathbf{0})\geqslant |p_n'(0)|\geqslant r^n$, i.e., (35) fails to hold.

It should be noted that in the above example $\mathbf{0}$ cannot be connected to the *interior* of $\mathbf{K}$ by a continuous curve. This phenomenon gives rise to the following

Conjecture. If $\mathbf{x}\in\mathbf{K}$ can be connected to the interior of $\mathbf{K}$ by a continuous curve, then (35) holds.

At present we cannot prove this conjecture, but we can point out a general class of domains (going beyond w-cuspidal sets) for which (35) is true. The cuspidal domains discussed in Theorems 1–3 above had the property that their cusps were located on algebraic curves contained in the domain. The methods of this paper can be extended to the study of more general domains which do not contain algebraic curves passing through the cusps, but possess algebraic curves located "sufficiently close" to the domain. For instance we could modify our definition of cusps by letting $\mathbf{q}$ to be any *continuous* curve but requiring at the same time that there exists a polynomial curve $\mathbf{m}$ such that

$$\operatorname{dist}(\mathbf{m}(t), \mathbf{q}(t))=o(w(t)). \tag{37}$$

It can be shown that (35) holds under this more general assumption (even though the exact order of magnitude of the MB-factor is hard to determine). For example the domain $\mathbf{K}=\{(x, y)\in\mathbf{R}^2: 0\leqslant x\leqslant 1,\ e^{-2/x}\leqslant y\leqslant e^{-1/x}\}$ has a cusp at the origin with no algebraic curve contained in $\mathbf{K}$ and passing through $\mathbf{0}$. On the other hand, (37) holds with $\mathbf{m}$ being the line segment $\{(x, 0): 0\leqslant x\leqslant 1\}$.

REFERENCES

1. M. Baran, Markov inequality on sets with polynomial parametrization, *Ann. Polon. Math.* **60** (1994), 69–79.
2. P. Borwein and T. Erdélyi, "Polynomials and Polynomial Inequalities," Springer, 1995.
3. T. Erdélyi, Remez-type inequalities on the size of generalized polynomials, *J. London Math. Soc.* **45** (1992), 255–264.
4. T. Erdélyi, A. Kroó, and J. Szabados, Markov–Bernstein type inequalities of compact subsets of **R**, *Analysis Math.*, to appear.

5. P. Goetgheluck, Inegalité de Markov dans les ensembles effilés, *J. Approx. Theory* **30** (1980), 149–154.
6. W. Pawłucki and W. Pleśniak, Markov's inequality and C^∞ functions on sets with polynomial cusps, *Math. Ann.* **275** (1986), 467–480.
7. W. Pleśniak, Again on Markov's inequality, *in* "Constructive Theory of Functions," pp. 679–683, Sofia, 1984.
8. V. Totik, Markoff constants for Cantor sets, *Acta Sci. Math. (Szeged)* **60** (1995), 715–734.